Creating and Using a Red-Black Matrix

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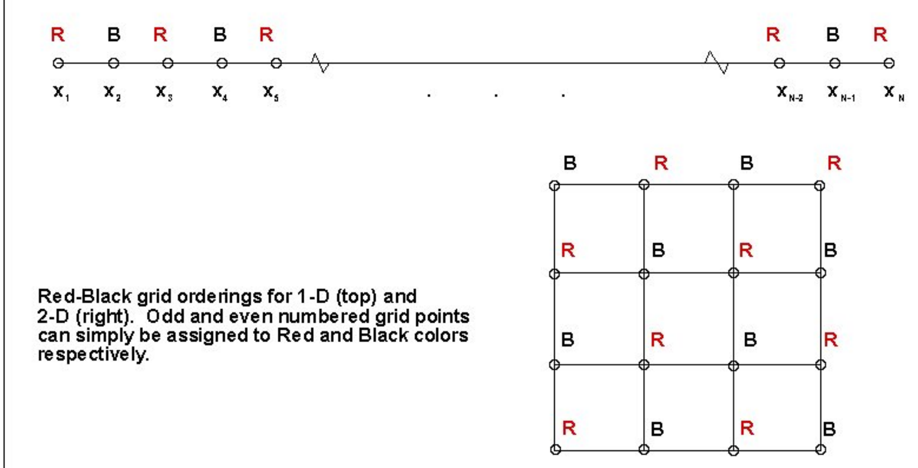
A red-black matrix is often useful when trying to compute an inherently sequential problem in parallel. For example, by using a red black coloring scheme for grid points, the Gauss-Seidel method can be vectorized to compute the solution on all Red grid points simultaneously, followed by all Black grid points. To create the red-black matrix, one should use a corresponding red-black grid which chooses alternating grid points in such a manner that odd grid points are colored red and even numbered grid points are colored black or vice-versa. The graph below shows a red-black ordering for 1D and 2D grids, this idea can also be extended similarly to 3D.



Red-Black grid orderings for 1-D (top) and 2-D (right). Odd and even numbered grid points can simply be assigned to Red and Black colors respectively.

A red-black matrix should take the block form

$$A = \begin{vmatrix} D_R & C^T \\ C & D_B \end{vmatrix}$$

Where D_1 and D_2 are strictly diagonal matrices and C is diagonally sparse. This would in general correspond to the solution of the linear system

$$\begin{vmatrix} D_R & C^T \\ C & D_B \end{vmatrix} \begin{vmatrix} x_R \\ x_B \end{vmatrix} = \begin{vmatrix} b_R \\ b_B \end{vmatrix}$$

Example 1: Converting a tridiagonal system to a red-black system

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{vmatrix} \quad \rightarrow \quad \begin{vmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_3 \\ b_2 \\ b_4 \end{vmatrix}$$

Example 2: Gauss-Seidel vectorized solution for the Red-Black matrix A

$$x_R^{n+1} = D_R^{-1}(b_R - C^T x_B^n)$$

$$x_B^{n+1} = D_B^{-1}(b_B - C^T x_R^{n+1})$$

Advanced References:

James W. Demmel. *Applied Numerical Linear Algebra* SIAM, 1997.

James Ortega. *Introduction to Parallel and Vector Solution of Linear Systems* Springer, 1988.